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NEARLY CONVEX SETS AND THE SHAPE OF LEGISLATIVE DISTRICTS

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Abstract: In this paper we examine how close a polygonal planar set is to being convex. This is accomplished by considering the ratio of the area of the largest convex set contained in the original polygon to the area of the convex hull of the set. Algorithms for determining the convex hull and for determining the largest convex set interior to the polygon are exhibited. After defining when such sets are *nearly convex* we then use this result to decide when legislative districts are *nicely shaped.*

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1. Introduction

The *convex hull* of a set of points in the plane is a well-defined mathematical object (see the next section for definitions). If given n points in the plane then there exist algorithms for finding the convex hull of these points in *n log n* time (see, e.g., Graham [4]). Note that the convex hull in these cases will be polygonal. A more recent problem is to find the largest convex polygon contained inside a given simple polygon. This is the *potato - peeling problem* of Goodman [3]. Such a polygon exists (see below for details) but this polygon is not well-defined. Goodman [3] gave a finite solution to this problem if the polygon has ≤ 5 sides. He also exhibited some properties of solutions to the problem. Chang and Yap [1] give an $O(n^7)$ time algorithm for finding this set, solving this problem computationally.

The redrawing of legislative districts in each state in the United States is an ongoing and controversial issue. Population numbers and the demographics of the population are taken into account in redistricting, as is the geography of the state. The political party in power at the time of the redistricting also tries to draw districts while taking into account the political affiliation of members of the district, to ensure that more members of the party in power are elected. This has lead to districts which meander around the state, are elongated and are very jagged. In general, they are not *nicely shaped* . In an attempt to take politics out of this redrawing, and to have districts that are as *nicely shaped* as possible, mathematical formulae have been developed to assign numerical values to district shapes. Generally, the closer to 1 this value is, the more *nicely shaped* it is. As this number gets closer and closer to zero, the uglier the shape becomes. Some of these tests (see descriptions below) include ones developed by Polsby-Popper [5], Schwartzberg [8] and Roeck [7]. All these measure what Schwartzberg [8] termed the *compactness* of the district. To a mathematician, of course, all of the districts are *compact* , but this terminology has stuck in the literature. All of these tests really speak to the *convexity* of a district and this is what our new test utilizes.

2. Definitions and Preliminary Results

We work throughout this paper in the Euclidean plane R^2.

Definition 1: *A* **simple polygonal region** *in the plane is a compact set in the plane bounded by a simple closed polygon.*

Definition 2: *A region (set of points) in the plane is said to be* **convex** *if every pair of points in the region can be connected by a straight line lying entirely in the region.*

Definition 3: *The* **convex hull** *or* **exogon** *of a simple polygonal region in the plane is the smallest convex set in the plane containing the polygon. (That this is a well-defined polygon is a well-known result.)*

Definition 4: *An* **endogon** *is a largest convex set contained in a simple polygonal region in the plane.*

That an endogon exists for every simple polygonal region in the plane follows from:

Proposition 1: *If C is a compact set in the plane, then C contains a closed convex subset of maximum area.*

Proof: This is a special case of Goodman's [3] Proposition 1 and the fact that a simple polygonal region is compact.

That an endogon is itself a simple polygonal region follows from:

Proposition 2: *If M is any maximal convex subset of a simple polygonal region in the plane, then M itself is a simple polygonal region.*

Proof: This is Goodman's [3] Proposition 3.

To see that an endogon is not well-defined, in particular not unique, consider a flattened hourglass, i.e., an orthogonal projection of a regular full cone situated in $\mathbf{R^3}$ with base on the xy plane, where the vertex is replaced by a thin cylinder connecting the top and bottom portions of the full cone, to the yz plane.

Current measures of the 'compactness' of legislative districts include:

Schwartzberg Test: The ratio of the legislative district's perimeter to its area. (Note that this value is not necessarily between 0 and 1.)

Polsby-Popper Test: The ratio of the area of the legislative district to the area A of a circle with circumference equal to the perimeter P of the district. This is given by the formula $(4\pi A)/(P^2)$ (verification left to the reader). This test corrects a problem with the Schwartzberg test (see below) and produces numbers between 0 and 1.

Roeck Test: The ratio of the area of the legislative district to the area of the smallest circle containing the district. This value is also between 0 and 1.

3. The Algorithms

There are many well known algorithms for finding the exogon, i.e. the convex hull, of a simple closed polygonal region. The best of these take $O(nlogn)$ time at worst, where n is the number of coordinates in the plane. One is by Graham [4], who was the first to achieve $nlogn$ time. In fact, a company associated with the University of California in Berkeley will take these n points and calculate the exogon, including its area.

An $O(n^7)$ time algorithm for finding the endogon can be found in Chang and Yap [1]. In this paper they show that the problem is finite, in general, and derive a polynomial time algorithm. They introduce the concept of a *balanced chain* in forming this algorithm and it turns out that this concept is the stumbling block when trying to find a faster algorithm (we are not aware of a quicker one).

We now introduce the ratio to be used in the sequel:

Definition 5: *A simple polygonal region P is said to be* **nearly convex** *if the ratio A_{end}/A_{exo} is $\geq .5$, where A_{end} is the area of the endogon of P and A_{exo} is the area of the exogon of P.*

4. The Shape of Legislative Districts

Numerous mathematical measures have been introduced which assign numbers to the shape of legislative districts in order to decide if the district is well-shaped or not. These tests give values as to the 'compactness' of a district.

This is not mathematical compactness, however, but the English definition of 'compact'. Some of these tests are mentioned above.

Each of these measures have difficulties. The Schwartzburg test gives varying values to districts, so it is hard to choose a numerical cut-off. Even worse, perfectly shaped districts yield different values. For instance, a square with sides 2 miles has a value of 2, while a square with sides 10 miles has a value of .4. This is certainly undesirable. The Roeck test has a different problem. It requires the area of the circumcircle of the district, but there are no algorithms for computing this, making it a hard test to administer.

The most popular and most used method is the Polsby-Popper test. It outputs values between 0 and 1. It is the ratio of the area of the district, which is known, to the area of the circle with circumference equal to the perimeter of the district, the latter value also being known. So it is easy to calculate and allows one to choose a cut-off value. However, this test does not always yield values corresponding to how nicely a district is shaped, since the perimeter can be arbitrarily long. This can be seen in the following figure and what follows:

Figure 1

The district on the left is clearly better shaped than the one on the right. Nonetheless, the Polsby- Popper test gives both these districts values $< .5$. In fact the district on the left has Polsby- Popper value .411 while the one on the right has value .337, quite close to one another. The values for these districts when the ratio of the area of their endogons to their exogons is computed are .898 and .449, respectively. (Note that these are approximations, albeit accurate ones.) We have:

Definition 5: *A legislative district is* **nicely shaped** *if it is nearly convex. Otherwise, it is* **poorly shaped.**

Then the figure on the left is nicely shaped, as expected, while the figure on the right is poorly shaped, again as expected.

One potential problem with this measure is that an extremely long and thin district can have a value of 1 in this model. While this may be desirable in some cases, to avoid this issue one could put a maximum length for any line lying in the district.

5. Conclusion

Many considerations must be taken into account when redrawing legislative districts. Some of these include the number of voters in the proposed district, the voters' ethnicity and their (supposed) shared interests. If the shape of the legislative district is to be a factor, and if a mathematical measure is to be employed to decide if a district is nicely shaped or not, then the ratio of the area of the endogon of the district to the area of the exogon of the district should be employed.

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7. References

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